Generation of the bosonic string theory solutions from the stationary Einstein fields via projection symmetry

Oleg V. Kechkin

DEPNI, Institute of Nuclear Physics,
M.V. Lomonosov Moscow State University,
Vorob'jovy Gory, 119899 Moscow, Russia,
e-mail: kechkin@depni.npi.msu.su

Abstract

A new formalism for generation of solutions in the consistently truncated low-energy bosonic string theory is developed. This formalism gives a correspondence of the projection type between the theories toroidally compactified from the diverse to three dimensions. Taking the stationary Einstein-dilaton gravity as the theory with a lower dimension, we generate its bosonic string theory extension and calculate the bosonic string theory solution corresponding to the Kerr-NUT one modified by the presence of the charged dilaton field.

1

1   Introduction

At low energies the bosonic string theory becomes a field theory of its massless modes [1]. These modes include the dilaton and Kalb-Ramond fields coupled to gravity. The effective gravity model lives in the multidimensional space-time and allows a toroidal compactification [2]. The resulting dynamical system possesses a rich group of hidden symmetries; in the case of the compactification to three dimensions this group coincides with $O(d, d)$, where d is the number of the compactified dimensions [3]. In fact $O(d, d)$ is the isometry group for the $O(d, d)/O(d) \times O(d)$ symmetric space model arising in this situation. The isometry transformations act in the solution space and give a base for the solution generation technique in the theory under consideration (see [4] for the Einstein and Einstein-Maxwell theory analogies and [5] for the string theory applications).

   In this article some special truncation of the bosonic string theory is considered [6]; the truncated theory is compactified to three dimensions on a torus. In fact such a procedure gives a series of the resulting 3-dimensional symmetric space models coupled to the effective 3-gravity: a concrete representative of this series is labeled by the number d. Our main goal is to establish a general map which relates the models with different numbers of the compactified dimensions. We obtain this map in the explicit form and use it in formulation of a technique for generation of the solutions for the theory with arbitrary value of d starting from the theory with $d = 2$. We show that the $d = 2$ theory coincides with the stationary Einstein gravity coupled to the dilaton field in view of our truncation. Thus, the proposed generation procedure realizes the general theory extension

$$\text{Stationary Einstein} - \text{Dilaton Theory} \rightarrow \text{Bosonic String Theory}, \tag{1.1}$$

which allows one to construct a lot of concrete solutions in he low-energy bosonic string theory. We take the Kerr-NUT solution [4] generalized by the presence of the charged dilaton field and calculate all the potentials related to this solution and necessary for our generation procedure. The resulting bosonic string theory solution describes a non-rotating point-like object with the nontrivial dilaton and Kalb-Ramond field multipole characteristics. In Conclusion we discuss the obtained results and give some natural following perspectives.

2   A model

We consider the effective field theory which describes the bosonic string theory massless modes living in the multidimensional space-time. Let $d + 3$, $(+ \cdots +; -, ++)$ and X^M be the space-time dimensionality, signature and the coordinates respectively $(M = 1, ..., d+3)$;

whereas Φ, B_{MN} and G_{MN} denote the dilaton, Kalb-Ramond and metric fields. The action representing a low-energy dynamics of these fields reads [1]:

$$S_{(d+3)} = \int dX \sqrt{-\det G_{MN}} e^{-\Phi} \left(R_{d+3} + \Phi_{,M} \Phi^{,M} - \frac{1}{12} H_{MNK} H^{MNK} \right), \qquad (2.1)$$

where $H_{MNK} = B_{NK,M} + B_{MN,K} + B_{KM,N}$. In this article we deal with some consistent truncation of this theory. To perform it, let us denote $y^m = X^m$ $(m = 1, ..., d)$ and $x^\mu = X^{d+\mu}$ $(\mu = 1, 2, 3)$. Then the truncation can be defined using a combination of the on-shell consistent restrictions

$$B_{mn} = G_{m,d+\mu} = B_{d+\mu,d+\nu} = 0 \qquad (2.2)$$

with the following toroidal compactification of the first d dimensions.

Thus, the remaining nontrivial field components depend on the coordinates x^μ and consist of the quantities G_{mn}, $B_{m,d+\mu}$ (we combine them to the $d \times d$ and $d \times 1$ matrices G and b_μ respectively), and also of the fields $G_{d+\mu,d+\nu}$ and Φ. Let us also put ([3])

$$\begin{aligned} h_{\mu\nu} &= e^{-2\phi} G_{d+\mu,d+\nu}, \\ \phi &= \Phi - \ln\sqrt{-\det G}; \end{aligned} \qquad (2.3)$$

below all the 3-dimensional operations will be related to the 3-metric $h_{\mu\nu}$ and its inverse. Then, let us introduce on shell of the motion equations the $d \times 1$ pseudo-scalar field v by help of the relation [3], [7]

$$\nabla v = e^{-2\phi} G^{-1} \nabla \times \vec{b}, \qquad (2.4)$$

where $(\vec{b})_\mu = b_\mu$. Then the dynamics of the 3-fields G, ϕ, v and $h_{\mu\nu}$ is given by the equations [6]

$$\begin{aligned} \nabla \vec{J} &= 0, \\ R_{3\,\mu\nu} &= \frac{1}{4} \mathrm{Tr} \left(J_\mu J_\nu \right), \end{aligned} \qquad (2.5)$$

where the Ricci tensor $R_{3\,\mu\nu}$ corresponds to the 3-metric $h_{\mu\nu}$, $\vec{J} = \nabla \mathcal{G} \, \mathcal{G}^{-1}$ and

$$\mathcal{G} = \begin{pmatrix} -e^{-2\phi} + v^T G v & v^T G \\ G v & G \end{pmatrix}. \qquad (2.6)$$

3

Also it is possible to introduce the vector matrix $\vec{\Omega}$, satisfying the relation

$$\nabla \times \vec{\Omega} = \vec{J}, \tag{2.7}$$

on shell of the system (2.5).

Now let us suppose that we have found a solution of the effective 3-dimensional problem under consideration, i.e. the quantities $\mathcal{G}$, $\vec{\Omega}$ and $h_{\mu\nu}$ satisfying Eq. (2.5) and (2.7). Then, as it follows from Eq. (2.6),

$$
\begin{aligned}
G &= \mathcal{G}_{22}, \\
e^{2\phi} &= -\left(\mathcal{G}^{-1}\right)_{11},
\end{aligned}
\tag{2.8}
$$

where the indexes correspond to the matrix blocks in Eq. (2.6). This completely defines the $(d+3)$-dimensional line element:

$$ds^2_{d+3} = dy^T G dy + e^{2\phi} ds^2_3, \tag{2.9}$$

where y is the $d \times 1$ column with the components y^m and $ds^2_3 = h_{\mu\nu} dx^\mu dx^\nu$. Then, as it is easy to prove, the remaining non-trivial $(d+3)$-dimensional fields are:

$$
\begin{aligned}
B_{m,d+\mu} &= -\left(\vec{\Omega}_{21}\right)_{m\mu}, \\
e^{\Phi} &= \sqrt{\det \mathcal{G}}\, e^{2\phi}.
\end{aligned}
\tag{2.10}
$$

Eqs. (2.8)-(2.10) translate a solution of the effective 3-dimensional system to the solution of the bosonic string theory (2.1).

3 A projection symmetry

The simplest solution of the low-energy bosonic string theory (2.1) satisfying the restrictions (2.2) describes the empty $(d+3)$-dimensional space-time. For this solution the corresponding 3-dimensional quantities are $\mathcal{G} = \mathcal{G}_0 = \mathrm{diag}(-1, -1; 1 \cdots 1)$, $\vec{\Omega} = 0$ and $h_{\mu\nu} = \delta_{\mu\nu}$. In this section we construct some symmetry extension $\mathcal{G}_0 \to \mathcal{G}$ of this trivial solution to the field of non-trivial ones of the form

$$\mathcal{G} = \mathcal{S}\mathcal{G}_0, \tag{3.1}$$

where $\mathcal{S}$ is the symmetry operator, and calculate corresponding quantities $\vec{\Omega}$ and $h_{\mu\nu}$. Our consideration, as well as the final results will have a form of projection of the original theory living in $d+3$ dimensions to the $(d'+3)$-dimensional one with $d' \leq d$.

Let us start with a set of the $(d+1) \times 1$ constant columns $\{C_k\}$ $(k = 1, ..., \mathcal{D})$ and consider the following $\mathcal{D}^2$ constant matrices:

$$\Pi_{kl} = C_k C_l^T \mathcal{G}_0. \tag{3.2}$$

Their mutual products read:

$$\Pi_{kl}\Pi_{mn} = \kappa_{lm}\Pi_{kn}, \tag{3.3}$$

where

$$\kappa_{lm} = \mathrm{Tr}\,\Pi_{ml} = C_m^T \mathcal{G}_0 C_l \tag{3.4}$$

is the set of the related constants. We name the matrices Π_{kl} "projectors" in view of Eq. (3.3): from this it follows that $\Pi_{kl}^2 \sim \Pi_{kl}$, i.e. the essentially projection property. From Eq. (3.1) one obtains another identity:

$$\Pi_{kl}^T = \mathcal{G}_0 \Pi_{lk} \mathcal{G}_0; \tag{3.5}$$

it relates the matrix and index symmetric properties of the projectors.

Now let us consider the index-free matrix

$$\Gamma = \xi_{kl}\Pi_{kl}, \tag{3.6}$$

where the non-matrix quantities ξ_{kl} are introduced and a summation over the repeating indexes is understood. Using Eqs. (3.3), (3.5) it is easy to prove that

$$\Gamma^n = \xi_{kl}^{(n)}\Pi_{kl}, \tag{3.7}$$

where

$$\xi^{(n)} = (\xi\kappa)^n \kappa^{-1}, \tag{3.8}$$

and the quantities ξ_{kl} and κ_{kl} are naturally embedded to the $\mathcal{D} \times \mathcal{D}$ matrices ξ and κ respectively; we also suppose that $\det \kappa \neq 0$. We define the symmetry operator as

$$\mathcal{S} = \exp \Gamma; \tag{3.9}$$

from Eqs. (3.7), (3.8) it follows that

$$\mathcal{S} = 1 + \left[(s-1)\kappa^{-1} \right]_{kl} \Pi_{kl}, \tag{3.10}$$

where $s = \exp\left(\xi\kappa\right)$. Thus, in the our construction ξ is the 3-dimensional matrix field. We need in knowledge of its algebraic and dynamic properties for complete definition of the symmetry operator $\mathcal{S}$.

The algebraic property immediately follows from symmetry of the matrix $\mathcal{G}$ (see Eq. (2.6)): this symmetry is guaranteed if $\mathcal{S}^T = \mathcal{G}_0 \, \mathcal{S} \, \mathcal{G}_0$, i.e. if

$$s^T = \kappa s \kappa^{-1} \tag{3.11}$$

in view of Eq. (3.10). From Eq. (3.11) it also follows that $\xi^T = \xi$. Then, the dynamical property of the matrix field s can be obtained from the supposition that the matrix $\mathcal{G}$ defined by Eqs. (3.1), (3.10) identically satisfies the motion equations (2.5). After some algebraical manipulations one concludes that them actually satisfied if

$$\begin{aligned} \nabla \vec{j}_s &= 0, \\ R_{3\,\mu\nu} &= \frac{1}{4}\mathrm{Tr}\left(j_{s\,\mu}j_{s\,\nu}\right), \end{aligned} \tag{3.12}$$

where $\vec{j}_s = \nabla s \, s^{-1}$ and the $\mathcal{G}$- and s-matrix currents are related as $\vec{J} = (\vec{j}_s\kappa^{-1})_{kl}\Pi_{kl}$. Finally, Eqs. (3.11) and (3.12) give the remaining part of information about the symmetry operator: it is defined in terms of a solution of the dynamical problem (3.12) restricted by Eq. (3.11). Also from Eq. (3.10) it follows that the trivial extension in Eq. (3.1) takes place if $s = 1$.

For us it will be useful to reformulate the problem for the matrix field s in the form similar to the one for the matrix $\mathcal{G}$. To do it, let us represent the symmetric matrix κ in the form of

$$\kappa = \tau^T g_0 \tau, \tag{3.13}$$

where g_0 is the signature matrix for κ. Then, let us introduce the following matrix field g:

$$g = g_0 \tau s \tau^{-1}. \tag{3.14}$$

It is easy to prove, that the field g satisfies the dynamical problem of the same type as the 'physical' matrix $\mathcal{G}$:

$$\begin{aligned} \nabla \vec{j} &= 0, \\ R_{3\,\mu\nu} &= \frac{1}{4}\mathrm{Tr}\left(j_\mu j_\nu\right), \end{aligned} \tag{3.15}$$

where $\vec{j} = \nabla g \, g^{-1}$; whereas the algebraical restriction coincides with the symmetric property:

$$g^T = g. \tag{3.16}$$

6

For the completeness of the g-formalism let us also introduce on-shell the vector matrix field $\vec{\omega}$ as

$$\nabla \times \vec{\omega} = \vec{j}. \tag{3.17}$$

Finally, we have reduced the $\mathcal{G}$-problem to the g-problem of the same type. From Eq. (3.14) it follows that $\mathcal{G} = \mathcal{G}_0$ if $g = g_0$. This means, in particular, that the map $g \to \mathcal{G}$ preserves the asymptotic flatness property of the fields if we consider the g-theory in the same manner as the theory related to the 'physical' matrix $\mathcal{G}$.

At the end of this section let us establish an explicit form of the relation between the g- and $\mathcal{G}$-theories. First of all, it is convenient to introduce the 'normalized' columns $C_{0\,k} = \left[(\tau^T)^{-1}\right]_{kl} C_l$. Second, let us combine the columns $C_{0\,k}$ to the single $(d+1) \times \mathcal{D}$ matrix C_0 in the natural way (so that $C_{0\,k}$ is the k-th column of C_0). Then from Eq. (3.4) one has:

$$\mathcal{C}^T \mathcal{G}_0 \mathcal{C} = g_0. \tag{3.18}$$

The resulting expressions for the $g \to \mathcal{G}$ map is remarkable κ-independent; after some algebra one concludes that

$$\begin{aligned}
\mathcal{G} &= \mathcal{G}_0 + \mathcal{C}\left(g_0 g g_0 - g_0\right)\mathcal{C}^T, \\
\mathcal{G}^{-1} &= \mathcal{G}_0 + \mathcal{C}\left(g^{-1} - g_0\right)\mathcal{C}^T, \\
\vec{\Omega} &= \mathcal{C} g_0 \vec{\omega} \mathcal{C}^T \mathcal{G}_0,
\end{aligned} \tag{3.19}$$

and also

$$\det \mathcal{G} = \det g_0 \, \det g. \tag{3.20}$$

Using the simple algebraical analysis, it is possible to show that Eq. (3.18) is compatible only if $\mathcal{D} \le d + 1$. In the case of exact equality one deals with application of the hidden symmetries related to our truncated bosonic string gravity model. Here the number $\mathcal{D}$ has a clear sense because $d = \mathcal{D} - 1$ is the number of toroidally compactified dimensions. It is possible to generalize this interpretation to the case of $\mathcal{D} < d + 1$: here we obtain a new type of symmetry extension of the theory with $d' = \mathcal{D} - 1$ compactified dimensions to the theory with $d > d'$ ones. It is easy to see that this new symmetry map cannot be generalized itself by the help of isometries acting inside of the g- and $\mathcal{G}$-theories if these isometries preserve a truncation conditions (2.2) for the both theories. Such generalizations act as the re-parameterizations of the matrix C_0 and preserve Eq. (3.18). At the end of this section let us note that the constructed new symmetry map has the explicitly projection type.

7

4   A generation procedure

Now it is possible to apply the decoding relations (2.8)-(2.10) to the projection scheme given in Eqs. (3.18)-(3.20). Let us introduce the $d \times d$ matrix $G_0 = \text{diag}(-1; 1, \cdots 1)$, which denotes the trivial G-value; let us also parameterize the projection matrix C_0 as

$$C_0 = \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}, \tag{4.1}$$

where the matrix blocks $\mathcal{A}$ and $\mathcal{B}$ are of the dimensions $1 \times \mathcal{D}$ and $d \times \mathcal{D}$ respectively. Then from Eq. (3.18) it follows that

$$\mathcal{B}^T G_0 \mathcal{B} - \mathcal{A}^T \mathcal{A} = g_0. \tag{4.2}$$

Then, application of Eqs. (2.8), (2.9) gives the explicit form of the $(d+3)$-dimensional line element,

$$ds_{d+3}^2 = dy^T \left[G_0 + \mathcal{B}(g_0^{-1} g g_0 - g_0) \mathcal{B}^T \right] dy + \left[1 - \mathcal{A}(g^{-1} - g_0) \mathcal{A}^T \right] ds_3^2, \tag{4.3}$$

whereas by the help of Eq. (2.10) one obtains the result for the $(d+3)$-dimensional dilaton and the non-trivial components of Kalb-Ramond fields:

$$e^\Phi = \sqrt{\det(g_0 g)} \left[1 - \mathcal{A}(g^{-1} - g_0) \mathcal{A}^T \right]$$
$$B_{m,d+\mu} = \left(\mathcal{B} \omega_\mu g_0 \mathcal{A}^T \right)_m. \tag{4.4}$$

Eqs. (4.3)-(4.4) define a solution of the bosonic string gravity system (2.1) truncated accordingly Eq. (2.2) in terms of the constant matrices $\mathcal{A}$ and $\mathcal{B}$ restricted by Eq. (4.2) and the matrix fields g and $\vec{\omega}$ satisfying Eqs. (3.15)-(3.17). This result takes place for the arbitrary $\mathcal{D} \leq d + 1$.

Now let us consider the special case of $\mathcal{D} = 2$ and $d \geq 2$, i.e. the essentially projection case (the bosonic string gravity with $d = 1$ was considered in [8]); let us also put $g_0 = 1$ for definiteness. Then the general solution of the algebraic equation (4.2) reads:

$$\mathcal{A} = (a_1 \; a_2), \qquad \mathcal{B} = \begin{pmatrix} b_1 & b_2 \\ \sqrt{1 + a_1^2 + b_1^2} \; n_1 & \sqrt{1 + a_2^2 + b_2^2} \; n_2 \end{pmatrix}, \tag{4.5}$$

where a_1, a_2; b_1, b_2 are the arbitrary parameters, whereas n_1 and n_2 are the $(d-1) \times 1$ columns. These columns satisfy the restrictions

$$n_1^T n_1 = n_2^T n_2 = 1,$$
$$n_1^T n_2 = \frac{a_1 a_2 + b_1 b_2}{\sqrt{1 + a_1^2 + b_1^2}\sqrt{1 + a_2^2 + b_2^2}} \tag{4.6}$$

and can be considered as the pair of unimodular $(d-1)$-dimensional vectors with some angle δ between them defined by the relation $\delta = \arccos\left(n_1^T n_2\right)$. For $d > 2$ the parameters a_1, a_2, b_1, b_2 are actually free because $(a_1 a_2 + b_1 b_2)/\sqrt{(1 + a_1^2 + b_1^2)(1 + a_2^2 + b_2^2)} \leq (a_1 a_2 + b_1 b_2)/\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} \leq 1$; this makes a definition of the angle δ through these parameters correct. In the case of $d = 2$ one has $\delta = 0, \pi$, and Eq. (4.6) can also be satisfied for some limit values of the discussing parameters.

Then, let us parameterize the dynamical matrices g and $\vec{\omega}$ in the following way:

$$g = e^{\psi} \Psi_4^{-1} \begin{pmatrix} \Psi_1 & \Psi_3 \\ \Psi_3 & \Psi_2 \end{pmatrix}, \qquad \vec{\omega} = \begin{pmatrix} \vec{\omega}_1 & \vec{\omega}_3 \\ \vec{\omega}_4 & \vec{\omega}_2 \end{pmatrix}, \tag{4.7}$$

where $\Psi_4^2 = \Psi_1 \Psi_2 - \Psi_3^2$ and the scalar and vector components are related accordingly Eq. (3.17). Let us denote $t = y^1$ (the time coordinate) and $Y^r = y^{r+1}$ (the extra dimension coordinates, $r = 1, ..., d-1$). Then for the $(d+3)$-dimensional line element one obtains that

$$ds_{d+3}^2 = -F(dt - K^T dY)^2 + dY^T H dY + E ds_3^2, \tag{4.8}$$

. where the quantities F, K, H and E read:

$$F = 1 + b_1^2 + b_2^2 - e^{\psi} \Psi_4^{-1} \left(b_1^2 \Psi_1 + b_2^2 \Psi_2 + 2 b_1 b_2 \Psi_3\right),$$

$$K = F^{-1} \left\{ \sqrt{1 + a_1^2 + b_1^2} \left[-b_1 + e^{\psi} \Psi_4^{-1} \left(b_1 \Psi_1 + b_2 \Psi_3\right)\right] n_1 + \sqrt{1 + a_2^2 + b_2^2} \left[-b_2 \right.\right.$$
$$\left.\left. + e^{\psi} \Psi_4^{-1} \left(b_2 \Psi_2 + b_1 \Psi_3\right)\right] n_2 \right\},$$

$$H = 1 + (1 + a_1^2 + b_1^2) \left\{-1 + e^{\psi} \Psi_4^{-1} \Psi_1 + F^{-1} \left[-b_1 + e^{\psi} \Psi_4^{-1} \left(b_1 \Psi_1 + b_2 \Psi_3\right)\right]^2\right\} n_1 n_1^T$$

$$+ (1 + a_2^2 + b_2^2) \left\{-1 + e^{\psi} \Psi_4^{-1} \Psi_2 + F^{-1} \left[-b_2 + e^{\psi} \Psi_4^{-1} \left(b_2 \Psi_2 + b_1 \Psi_3\right)\right]^2\right\} n_2 n_1^2$$

$$+ \sqrt{(1 + a_1^2 + b_1^2)(1 + a_2^2 + b_2^2)} \left\{e^{\psi} \Psi_4^{-1} \Psi_3 + F^{-1} \left[-b_1 + e^{\psi} \Psi_4^{-1} \left(b_1 \Psi_1 + b_2 \Psi_3\right)\right]\right.$$

$$\left. \times \left[-b_2 + e^{\psi} \Psi_4^{-1} \left(b_2 \Psi_2 + b_1 \Psi_3\right)\right]\right\} \left(n_1 n_2^T + n_2 n_1^T\right),$$

$$E = 1 + a_1^2 + a_2^2 - e^{-\psi} \Psi_4^{-1} \left(a_2^2 \Psi_1 + a_1^2 \Psi_2 - 2 a_1 a_2 \Psi_3\right); \tag{4.9}$$

9

whereas the non-zero components of the $(d+3)$-dimensional matter fields are

$$e^{\Phi} = e^{\psi}\left(1 + a_1^2 + a_2^2\right) - \Psi_4^{-1}\left(a_2^2\Psi_1 + a_1^2\Psi_2 - 2a_1a_2\Psi_3\right),$$
$$B_{t,d+3} = (a_1\omega_1 + a_2\omega_3)b_1 + (a_2\omega_2 + a_1\omega_4)b_2,$$
$$B_{r,d+3} = \sqrt{1 + a_1^2 + b_1^2}(a_1\omega_1 + a_2\omega_3)n_{1\,r} + \sqrt{1 + a_2^2 + b_2^2}(a_2\omega_2 + a_1\omega_4)n_{2\,r}. \quad (4.10)$$

Eqs. (4.8)-(4.10) define the generation procedure completely; the generation parameters are restricted accordingly Eq. (4.6). It is easy to see that this procedure possesses a wide group of symmetry transformations itself. Actually, the restrictions (4.6) on the columns n_1, n_2 preserve the form if one rotates the column pair 'as the solid body' in the effective $(d-1)$-dimensional parametric space. Then, it is possible to combine the parameters a_1, a_2, b_1, b_2 to the 2-dimensional vectors (a_1, b_1) and (a_2, b_2). It is clear that the right hand side of the second equation in (4.10) is invariant in respect to rotations in this space. Finally, the full parametric symmetry group of the our generation procedure is $O(d-1) \times O(2)$. This symmetry group acts as the group of re-parameterizations of the numeric parameters of the solution; in applications it can be convenient to use it for the 'gauge fixation' or, equivalently, for the concrete choice of frame in the effective parametric space of the solution.

5 A solution

Let us consider the generation base, i.e., the theory related to the dynamical matrices g and $\vec{\omega}$. Let us put

$$g = e^{\psi}\hat{g}; \qquad (5.1)$$

then from Eq. (4.7) it follows that $\det\hat{g} = 1$, i.e. $\hat{g} \in SL(2,R)/SO(2)$. In terms of the parameterization (5.1) the motion equations (3.15) take the form

$$\begin{aligned}
\nabla\hat{\vec{j}} &= 0, \qquad \nabla^2\psi = 0, \\
R_{3\,\mu\nu} &= \frac{1}{4}\text{Tr}\left(\hat{j}_\mu\hat{j}_\nu\right) + \frac{1}{2}\psi_{,\mu}\psi_{,\nu},
\end{aligned} \qquad (5.2)$$

where $\hat{\vec{j}} = \nabla\hat{g}\,\hat{g}^{-1}$. Then, for the matrix $\vec{\omega}$ one has the following decomposition:

$$\vec{\omega} = \hat{\vec{\omega}} + \tilde{\vec{\omega}}, \qquad (5.3)$$

where $\hat{\vec{\omega}} = \hat{\vec{j}}$ and $\tilde{\vec{\omega}} = \nabla\psi$.

The system under consideration is equivalent on-shell to the stationary 4-dimensional Einstein-dilaton theory, here the matrix $\hat{g}$ is constructed from the pure Einstein variables, whereas ψ denotes the effective dilaton field. This interpretation is alternative to the one given above: there the model related to the matrix field g was the 4-dimensional bosonic string theory truncated accordingly Eq. (2.2). In framework of the new interpretation the variables Ψ_α, $\alpha = 1, ..., 4$ are related to the Ernst representation of the stationary Einstein gravity as

$$\mathcal{E} = \Psi_1^{-1} \left(\Psi_4 + i\Psi_3 \right), \tag{5.4}$$

where $\mathcal{E}$ is the Ernst potential [9]. Our goal is to use a solution space of the General Relativity for calculation of the functions Ψ_α and $\vec{\omega}_\alpha$ which define the $(d+3)$-dimensional fields of the bosonic string theory accordingly Eqs. (4.8)-(4.10). To do it, let us suppose that the problem (5.2) is axisymmetric, i.e. the fields are φ-independent in the canonic Weil coordinates ρ, z, φ [4]. In these coordinates the 3-dimensional line element reads:

$$ds_3^2 = \exp\left(\gamma\right) \left(d\rho^2 + dz^2 \right) + \rho^2 d\varphi^2; \tag{5.5}$$

the $\hat{g}$- and ψ-equations in (5.2) are γ-independent and, thus, they defined over the flat 3-space. Then,

$$\gamma = \hat{\gamma} + \tilde{\gamma}, \tag{5.6}$$

where the functions $\hat{\gamma}$ and $\tilde{\gamma}$ are defined by the Einstein part of Eq. (5.2) for the case of trivial value of ψ and $\hat{g}$ respectively. In this scheme the $\tilde{\gamma}$-problem reads:

$$\begin{aligned}
\tilde{\gamma}_{,z} &= \psi_{,\rho}\psi_{,z} \\
\tilde{\gamma}_{,\rho} &= \frac{1}{2}(\psi_{,\rho}^2 - \psi_{,z}^2).
\end{aligned} \tag{5.7}$$

Let us also note that in the axisymmetric case the vector quantities $\hat{\vec{\omega}}$ and $\tilde{\vec{\omega}}$ have the non-zero φ-components only. Finally, to obtain the base for generation accordingly Eqs. (4.8)-(4.10), one must take a solution of the Einstein problem and to calculate the potentials $\Psi_\alpha, \hat{\omega}_{\alpha,\varphi}, \hat{\gamma}$ and also a solution ψ of the Laplace equation together with the related functions $\tilde{\omega}_\alpha, \tilde{\gamma}$. As it was shown above, to obtain asymptotically flat $(d+3)$-dimensional solution one must start from solution of the effective Einstein-dilaton theory possessing the same property.

Below we give the expressions for the all necessary quantities in terms of the coordinates R, θ, φ which relate to the canonical Weil ones as

$$\begin{aligned}
\rho &= \sqrt{R^2 + \alpha^2}\sin^2\theta, \\
z &= R\cos\theta.
\end{aligned} \tag{5.8}$$

11

We consider the class of Kerr-NUT solutions of Einstein theory coupled to the generalized Coulomb solution for the effective dilaton field. The corresponding Ψ_α-potentials are:

$$
\begin{aligned}
\Psi_1 &= (R + m_1)^2 + (a\cos\theta - m_2)^2, \\
\Psi_2 &= (R - m_1)^2 + (a\cos\theta + m_2)^2, \\
\Psi_3 &= -2(m_2 R + m_1 a\cos\theta), \\
\Psi_4 &= R^2 + a^2\cos^2\theta - |m|^2;
\end{aligned} \tag{5.9}
$$

whereas the potential ψ reads:

$$
\psi = \frac{2(q_1 - q_2\alpha\cos\theta)}{R^2 + \alpha^2\cos^2\theta}. \tag{5.10}
$$

Here $m = m_1 + im_2$ and $q = q_1 + iq_2$ are the arbitrary complex constants, a is the real parameter and $\alpha = \sqrt{a^2 - |m|^2}$. Then, the potentials $\omega_{\alpha,\varphi}$ are:

$$
\omega_{1\varphi} = 2(q_1 + m_1)\cos\theta \;+\; 2\sin\theta^2\left[\frac{\alpha(q_2 R + q_1\alpha\cos\theta)}{R^2 + \alpha^2\cos\theta 2} + \frac{a(m_2 R + m_1 a\cos\theta)}{\Psi_4}\right],
$$

$$
\omega_{2\varphi} = 2(q_1 - m_1)\cos\theta \;+\; 2\sin\theta^2\left[\frac{\alpha(q_2 R + q_1\alpha\cos\theta)}{R^2 + \alpha^2\cos\theta 2} - \frac{a(m_2 R + m_1 a\cos\theta)}{\Psi_4}\right],
$$

$$
\omega_{3\varphi} = -2m_2\cos\theta \;+\; 2\sin\theta^2\frac{a(m_1 R - m_2 a\cos\theta + |m|^2)}{\Psi_4},
$$

$$
\omega_{4\varphi} = -2m_2\cos\theta \;+\; 2\sin\theta^2\frac{a(m_1 R - m_2 a\cos\theta - |m|^2)}{\Psi_4}. \tag{5.11}
$$

Finally, the 3-metric for this solution is:

$$
\begin{aligned}
ds_3^2 \;=\;& \exp\left(-\frac{\sin^2\theta}{2}\left\{\frac{q_1^2 + q_2^2}{R^2 + \alpha^2\cos^2\theta} + \frac{R^2 + \alpha^2}{(R^2 + \alpha^2\cos^2\theta)^4}\left[(q_1^2 - q_2^2)(R^4 - 6R^2\alpha^2\cos^2\theta\right.\right.\right. \\
&+\; \left.\left.\left. \alpha^4\cos^4\theta) - 8q_1 q_2 R\alpha(R^2 - \alpha^2\cos^2\theta)\cos\theta\right]\right\}\right)\Psi_0\left(\frac{dR^2}{R^2 + \alpha^2\cos^2\theta} + d\theta^2\right) \\
&+\; (R^2 + \alpha^2\cos^2\theta)\sin^2\theta\, d\varphi^2.
\end{aligned} \tag{5.12}
$$

Eqs. (4.8)-(4.10) and (5.9)-(5.12) completely define our class of solutions for the $(d + 3)$-dimensional low energy bosonic string theory. This class depends on the set of $2d + 7$ parameters restricted by 3 relations, i.e. the constructed solution class has $2d + 4$ parametric degrees of freedom.

At the end of this section let us discuss our choice of solution for the generation base, i.e. for the effective Einstein-dilaton theory. Our statement is that for the solution taken the Einstein and dilaton parts have the same formal nature. Actually, let us calculate the potential $\zeta = (1 - \mathcal{E})/(1 + \mathcal{E})$ ([10]) for the Kerr-NUT solution; the result reads:

$$\zeta = \frac{m}{R - ia\cos\theta}. \tag{5.13}$$

Then, let us consider the complex solution of the axisymmetric Laplace equation given by the function ξ,

$$\xi = \frac{q}{R - i\alpha\cos\theta}; \tag{5.14}$$

it is seen that $\zeta \to \xi$ if $m \to q$ and $a \to \alpha$. Then our statement is related to the fact that $\psi = \xi + \bar{\xi}$.

Acknowledgments

6 Conclusion

In this article we have established a general form of the relation between the solution spaces of the effective field theories resulting from the toroidal compactification of the low-energy bosonic string theory from the diverse to three dimensions. Using this relation, we have developed a technique for generation of new solutions for the theory living in the space-time of the arbitrary dimensionality starting from the stationary 4-dimensional Einstein-dilaton gravity. We have calculated the special solution which describes a new bosonic string theory extension of the Kerr-NUT solution modified by the presence of the dilaton field of the Coulomb type. This concrete solution is not of the form known in the literature: it is static from the point of view of the physical 3-dimensional space. Nevertheless, this solution possesses the rotational properties, but its rotation related to the extra dimensions. The direction of a rotational axis in the extra dimensions is defined by the pair of the unimodular vectors n_1 and n_2, and depends on the position in the physical 3-space. The constructed solution is charged and possesses the peculiarities of the Dirac type for the Kalb-Ramond field components. These peculiarities can be removed using some special choice of the free solution parameters; we leave the following physical analysis and discussion to the future.

The perspectives are related to the possible generalizations of the obtained results to the case of the non-constrained bosonic string gravity and, moreover, to the case of the gravity model originated from the low-energy heterotic string theory. The corresponding gravity model generalizes the one considered above by the presence of some number of the $U(1)$ gauge fields. In [11], [12] for such gravity model with the arbitrary number of the Abelian gauge fields and toroidally compactified to three dimensions it was developed a formalism which seems really promising for the solution of the discussing problems. Moreover, there is a hope to generalize the projection formalism and the corresponding generation procedure to the case of arbitrary theory possessing a symmetric space model representation after the compactification to three dimensions [13]. Also the important perspective is related to formulation of the generation procedure based on the stationary Einstein-Maxwell theory. Its realization allows one to extend the widest solution space of the Einstein-Maxwell theory to the case of weakly studied field theory limits of the superstring theory. Coming back to the symbolic formula (1.1), let us represent the future perspectives in the following form:

$$\text{Stationary Einstein} - \text{Maxwell Theory} \rightarrow \text{Heterotic String Theory}. \qquad (6.1)$$

Of course, the natural and the nearest application of a such formalism is the generation of the heterotic string theory extension of the Kerr-Newman solution with the nontrivial magnetic and NUT charges. The first step in this direction had been performed in ([14]), where the special case of the 4-dimensional heterotic string theory with two Abelian gauge fields was considered. This special case was considered in the literature in view of its relation to the $D = N = 4$ supergravity (see [15], where the supergravity generalization of the Israel-Wilson-Perjes solution of Einstein-Maxwell theory was obtained and also [16] for the black hole solution construction in supergravity using harmonic functions). In ([14]) it was shown that for this special heterotic string gravity model the generation program can be realized in terms of some single complex 2×2 matrix potential which transforms linearly under the action of symmetries preserving the asymptotic flatness property of the fields. For the general case of the arbitrary original space-time dimensionality and number of the gauge fields it seems the most natural to use a language of the real matrix generalization of the ζ-potential developed for the heterotic string gravity in [12]. A solution of the discussing problems is now in progress.

References

[1] E. Kiritsis, "Introduction to superstring theory", Leuven Univ. Pr. (1998).

[2] J. Maharana, J.H. Schwarz, Nucl. Phys. **B390** (1993) 3.

[3] A. Sen, Nucl Phys **B434** (1995) 179.

[4] D. Kramer, H. Stephani, M. McCallum, E. Herlt, "Exact solutions of the Einstein field equations", Deutcher Verlag der Wissenschaften, Berlin, (1980).

[5] D. Youm, Phys. Rept. **316** (1999) 1.

[6] A. Herrera-Aguilar, O.V. Kechkin, Mod. Phys. Lett. **A16** (2001) 29.

[7] A. Herrera-Aguilar, O.V. Kechkin, Int. J. Mod. Phys. **A13** (1998) 393.

[8] I. Bakas, Nucl. Phys. **B428** (1994) 374.

[9] F. Ernst, Phys. Rev. **167** (1968) 1175.

[10] P.O. Mazur, Acta Phys. Pol. **B14** (1983) 219.

[11] A. Herrera-Aguilar, O.V. Kechkin, Phys. Rev. **D59** (1999) 124006.

[12] O.V. Kechkin, "New matrix formalism for heterotic string theory on a torus" gr-qc/0012090.

[13] P. Breitenlohner, D. Maison, G.W. Gibbons, Commun. Math. Phys. **120** (1988) 295.

[14] O.V. Kechkin, "Generation of heterotic string theory solutions from the stationary Einstein-Maxwell fields", submitted to Phys. Lett. **B**.

[15] E. Bergshoeff, R. Kallosh, T. Ortin, Nucl. Phys. **B478** (1996) 156.

[16] W.A. Sabra, Nucl. Phys. **B510** (1998) 247.